continuation of Vector Spaces 2

$$\mathbb{P}_2(\mathbb{R}) = \left\{ p(x) \in \mathbb{P}_2 \,/\, p(x) = ax^2 + bx + c \quad \forall a,b,c \in \mathbb{R} \right\}$$

$$\mathbb{R}^3(\mathbb{R}) = \left\{ \bar{p} \in \mathbb{R}^3 \,/\, \bar{p} = (a, b, c) \quad \forall a,b,c \in \mathbb{R} \right\}$$

$$B = \{ \underset{(1,0,0)}{x^2}, \underset{(0,1,0)}{x}, \underset{(0,0,1)}{1} \}$$

$$B' = \{ \underset{(1,1,1)}{x^2+x+1}, \underset{(1,1,0)}{x^2+x}, \underset{(1,0,0)}{x^2} \}$$

$$F_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\boxed{F_{B'} \; ?}$$

Change of base for an endomorphism matrix

$$V \xrightarrow{f} V \; + \; B \underset{C^{-1}}{\overset{C}{\rightleftarrows}} B' \Rightarrow F_{B'} = C^{-1} F_B C$$

$$C = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} x^2+x+1 & x^2+x & x^2 \end{array} \end{array} \qquad C^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

expressed in B

$$F_{B'} = C^{-1} F_B C = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{C^{-1}F_B} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{F_{B'}}$$

Working in $\mathbb{R}^3(\mathbb{R})$

We have an ENDOMORPHISM $f(x^1, x^2, x^3) = (x^1+x^2+x^3,\ x^1+x^2+x^3,\ x^1+x^2+x^3)$

We have a base $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and another base $B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ where $\begin{cases} \bar{u}_1 = \bar{e}_1 + \bar{e}_2 + \bar{e}_3 \\ \bar{u}_2 = \bar{e}_1 + \bar{e}_2 \\ \bar{u}_3 = \bar{e}_1 \end{cases}$

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

Matrix operation works in the opposite direction to this

$$B \underset{C^{-1}}{\overset{C}{\rightleftarrows}} B'$$

$$F_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\bar{u}_1\ \bar{u}_2\ \bar{u}_3$ expressed in B

$$\bar{v} = (1, 2, 3)_B = \bar{e}_1 + 2\bar{e}_2 + 3\bar{e}_3$$

$$F_{B'} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}}_{C^{-1}} \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B}_{\bar{v}} = \underbrace{\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}_{B'}}_{\bar{v}}$$

$$f(\bar{v}) = \bar{w}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{F_B} \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B}_{\bar{v}} = \underbrace{\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}_B}_{\bar{w}}$$

$$\bar{v} = (3, -1, -1)_{B'} = 3\bar{u}_1 - \bar{u}_2 - \bar{u}_3$$

$$\underbrace{\begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{F_{B'}} \underbrace{\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}_{B'}}_{\bar{v}} = \underbrace{\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}_{B'}}_{\bar{w}}$$

Making sure it all works

$$\boxed{\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}_{B'}}_{\bar{w}} = \underbrace{\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}_B}_{\bar{w}}}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}_{B'}}_{\bar{v}} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B}_{\bar{v}}$$

Vector Spaces 3

Given a homomorfism from $\mathbb{R}^3$ to $\mathbb{R}^2$ that works $f(x^1,x^2,x^3)=(x^1+x^3,\ x^2+x^3)$, calculate:

1. Ker(f) and Im(f) (giving base and dimension).
2. Calculate two bases from $\mathbb{R}^3$ and $\mathbb{R}^2$ so that f in those bases has the expression $\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$

$$\left.\begin{array}{l} B_1=\{\bar{e}_1,\bar{e}_2,\bar{e}_3\}\subset\mathbb{R}^3\\ B_2=\{\bar{u}_1,\bar{u}_2\}\subset\mathbb{R}^2\end{array}\right\}\text{ Bases of reference for the homomorphism}$$

$$f:\mathbb{R}^3\longrightarrow\mathbb{R}^2$$

$$\boxed{\begin{array}{ll}\bar{e}_1=(1,0,0)_{B_1} & \bar{u}_1=(1,0)_{B_2}\\ \bar{e}_2=(0,1,0)_{B_1} & \bar{u}_2=(0,1)_{B_2}\\ \bar{e}_3=(0,0,1)_{B_1} & \end{array}}$$

$$f(\bar{e}_1)=(1+0,0+0)=(1,0)_{B_2}$$
$$f(\bar{e}_2)=(0+0,1+0)=(0,1)_{B_2}$$
$$f(\bar{e}_3)=(0+1,0+1)=(1,1)_{B_2}$$

$$F_{B_1B_2}=\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}$$

$f(\bar{e}_1)$ $f(\bar{e}_2)$ $f(\bar{e}_3)$ expressed in B_2

$$\dim(\mathrm{Im}(f))=Rg(F_{B_1B_2})=2=\dim(\mathbb{R}^2)\longrightarrow \mathrm{Im}(f)=\mathbb{R}^2\longrightarrow B_{Im}=B_2=\{\bar{u}_1,\bar{u}_2\}$$

$$\underbrace{\dim(\mathrm{Im}(f))}_{2}+\dim(\mathrm{Ker}(f))=\underbrace{\dim(\mathbb{R}^3)}_{3}\longrightarrow \dim(\mathrm{Ker}(f))=1$$

$$\mathrm{Ker}(f):\ \underbrace{\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}}_{F_{B_1B_2}}\underbrace{\begin{pmatrix}x^1\\x^2\\x^3\end{pmatrix}_{B_1}}_{\bar{x}}=\underbrace{\begin{pmatrix}0\\0\end{pmatrix}_{B_2}}_{\bar{0}}\longrightarrow\begin{cases}x^1+x^3=0\\x^2+x^3=0\end{cases}$$

$$\begin{cases}x^1=-\gamma\\x^2=-\gamma\\x^3=\gamma\end{cases}\forall\gamma\in\mathbb{R}\qquad B_{Ker}=\{(-1,-1,1)\}$$

2. $F_{B_1B_2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$f(\bar{e}_1)$ $f(\bar{e}_2)$ $f(\bar{e}_3)$ expressed in B_2

$F_{B_3B_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$f(\bar{a}_1)$ $f(\bar{a}_2)$ $f(\bar{a}_3)$ expressed in B_4

$f(\bar{a}_1) = \bar{b}_1$
$f(\bar{a}_2) = \bar{b}_2$
$f(\bar{a}_3) = \bar{0} \longrightarrow \bar{a}_3 \in Ker(f)$

$\bar{e}_1 = (1,0,0)_{B_1}$ $\bar{u}_1 = (1,0)_{B_2}$
$\bar{e}_2 = (0,1,0)_{B_1}$ $\bar{u}_2 = (0,1)_{B_2}$
$\bar{e}_3 = (0,0,1)_{B_1}$

$B_3 = \{\bar{a}_1, \bar{a}_2, \bar{a}_3\}$?

$B_4 = \{\bar{b}_1, \bar{b}_2\}$?

Conditions that B_3 and B_4 must have:

1. $\bar{a}_1, \bar{a}_2$ and $\bar{a}_3$ must be L.I.
2. $\bar{b}_1$ and $\bar{b}_2$ must be L.I.
3. $\bar{a}_3 \in Ker(f)$ → MOST RESTRICTIVE
4. $\bar{a}_1$ and $\bar{a}_2$ transform into L.I. vectors.

$B_{Ker} = \{(-1,-1,1)\}$

We start off with 3 and establish:

$\bar{a}_3 = (-1,-1,1)_{B_1}$ for example

So now we pick an $\bar{a}_1$ and $\bar{a}_2$ L.I. with $\bar{a}_3$ to cover 1:

$\bar{a}_1 = (1,0,0)_{B_1}$
$\bar{a}_2 = (0,1,0)_{B_1}$

$\begin{vmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0 \rightarrow$ they are L.I.

We now transform $\bar{a}_1$ and $\bar{a}_2$ to see if the become L.I. vector and cover 2 and 4:

$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{F_{B_1B_2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{B_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{B_2} \longrightarrow \bar{b}_1 = (1,0)_{B_2}$

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \rightarrow$ they are L.I.

$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{F_{B_1B_2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{B_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{B_2} \longrightarrow \bar{b}_2 = (0,1)_{B_2}$

$\bar{a}_1 = (1,0,0)_{B_1}$
$\bar{a}_2 = (0,1,0)_{B_1}$
$\bar{a}_3 = (-1,-1,1)_{B_1}$
$\bar{b}_1 = (1,0)_{B_2}$
$\bar{b}_2 = (0,1)_{B_2}$